# Rationality does not specialize among terminal varieties

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An algebraic variety is *rational* if it becomes isomorphic to projective space after removing lower-dimensional subvarieties from both sides. Little is known about how rationality behaves in families. In particular, given a family of projective varieties for which the geometric generic fiber is rational, is every fiber geometrically rational? ("Geometric" refers to properties of a variety after extending its base field to be algebraically closed.)

Matsusaka proved that the analogous question for geometric ruledness has a positive answer [5, Theorem IV.1.6]. (By definition, a variety is ruled if it is birational to the product of the projective line with some variety.) That is, ruledness specializes in families of varieties. For example, Kollár used Matsusaka's theorem to show that a large class of Fano hypersurfaces are not ruled and therefore not rational [5, Theorem V.5.14]. By contrast, rationality does not specialize in this generality, as shown by a family of cubic surfaces over the complex numbers $\mathbf{C}$ with most fibers smooth and one fiber the projective cone over a smooth cubic curve. Every smooth cubic surface is rational, but the cone over a smooth cubic curve $E$ is birational to $E \times \mathbf{P}^1$, which is not rational because it has a nonzero holomorphic 1-form.

Note, however, that the cone over a cubic curve has a fairly bad singularity: it is log canonical but not klt (Kawamata log terminal). This suggests the question of whether rationality specializes among varieties with milder singularities. Indeed, it follows from de Fernex and Fusi [2, Theorem 1.3] and Hacon and M$^{c}$Kernan [3, Corollary 1.5] that rationality specializes among klt complex varieties of dimension at most 3.

Extending work of Voisin [9] and Colliot-Thélène and Pirutka [1], [8, Theorem 2.1] showed that a large class of Fano hypersurfaces $X$ are not stably rational. (That is, no product of $X$ with projective space is rational.) As an application, suggested by de Fernex, [8, Corollary 4.1] showed that rationality does not specialize among klt varieties of dimension 4 or higher.

In this paper, we find that the results of [8] are strong enough to imply that rationality does not specialize even among *terminal* varieties. Terminal singularities form the narrowest class of singularities that comes up in the minimal model program. The examples are in any dimension at least 5.

Some natural remaining questions are: Does rationality specialize among terminal 4-folds? Does rationality specialize among smooth varieties?

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## 1 The example

**Theorem 1.1.** *There is a flat projective morphism $f\colon X \to C$ with $C$ a Zariski open subset of the complex affine line such that $0$ is in $C$, all fibers of $f$ have terminal singularities, all fibers of $f$ over $C - 0$ are rational, and the fiber $F$ over $0$ is not rational.*

*Such examples exist with $F$ of any dimension at least 5. There is also a family of 4-folds with canonical singularities over a Zariski open subset $C$ of $A^1_{\mathbf{C}}$ such that all fibers over $C - 0$ are rational and the fiber $F$ over $0$ is not rational.*

In other words, rationality does not specialize among terminal varieties of dimension at least 5, or among canonical varieties of dimension at least 4. (Throughout, we are talking about families of projective varieties.)

*Proof.* (Theorem 1.1)

We start with the following old observation.

**Lemma 1.2.** *If $X$ is a hypersurface of degree $d$ in $\mathbf{P}^{n+1}$ over a field $k$ such that $X$ has multiplicity equal to $d-1$ at some $k$-rational point $p$, and if the singular locus of $X$ has codimension at least 2, then $X$ is rational over $k$.*

*Proof.* The assumption on the singular locus ensures that $X$ is irreducible. The assumption on the multiplicity of $X$ at $p$ implies that a general line through $p$ meets $X$ in exactly one other point. That gives a birational map over $k$ from the projective space $\mathbf{P}^n$ of lines through $p$ to $X$. □

We return to the proof of Theorem 1.1. By [8, Theorem 2.1], a very general quartic 4-fold in $\mathbf{P}^5_{\mathbf{C}}$ is not stably rational. Choose one smooth quartic 4-fold $Y$ over $\mathbf{C}$ which is not stably rational. Let $X_0$ be the projective cone over $Y$ in $\mathbf{P}^6$. Then $X_0$ is a quartic 5-fold, and $X_0$ is not rational because it is birational to $\mathbf{P}^1 \times Y$. Also, $X_0$ is terminal, because $Y$ has Fano index 2 which is greater than 1, meaning that the anticanonical bundle $-K_Y$ is given by $-K_Y \cong -(K_{\mathbf{P}^5}+Y)|_Y = O(6-4)|_Y = O(2)|_Y$ [6, Lemma 3.1].

Let $Y$ be defined by the equation $f_4(x_0, \ldots, x_5) = 0$. Then $X_0$ is defined by the same equation in $\mathbf{P}^6 = \{[x_0, \ldots, x_6]\}$. Let $g_3(x_0, \ldots, x_5)$ be a nonzero cubic form over $\mathbf{C}$. Consider the pencil of quartics in $\mathbf{P}^6$ given by the equation

$$f_4(x_0, \ldots, x_5) + a g_3(x_0, \ldots, x_5) x_6 = 0$$

for $a$ in the affine line $A^1_{\mathbf{C}}$. This gives a flat family $f\colon X \to A^1$ of hypersurfaces, and the fiber over 0 is the cone $X_0$. Since "terminal" is a Zariski-open condition in families [7, Corollary VI.5.3], there is a Zariski open neighborhood $C$ of 0 in $A^1$ such that all fibers of the restricted family $f\colon X_C \to C$ are terminal. In particular, the fibers are normal and hence have singular locus of codimension at least 2.

Finally, for all $a \neq 0$ in $C$, the fiber $X_a$ is a hypersurface of degree 4 in $\mathbf{P}^6$ with multiplicity equal to 3 at the point $[0, \ldots, 0, 1]$. By Lemma 1.2, it follows that $X_a$ is rational for all $a \neq 0$ in $C$. Since $X_0$ is not rational, this completes the proof that rationality does not specialize among terminal varieties.

The example given is a family of 5-folds. Multiplying the family with any projective space $\mathbf{P}^m$ shows that rationality does not specialize among terminal varieties of

any dimension at least 5. (Here again, it is important that $Y$ is not stably rational, so that $X_0 \times \mathbf{P}^m$ is not rational.)

Finally, replace the 4-fold $Y$ by a smooth quartic 3-fold (again called $Y$) in $\mathbf{P}^4_{\mathbf{C}}$ which is not stably rational. Such a variety exists, by Colliot-Thélène and Pirutka [1]. It follows that the projective cone $X_0$ over $Y$ in $\mathbf{P}^5$ (rather than $\mathbf{P}^6$) is not rational. Since $Y$ has Fano index 1, $X_0$ has canonical but not terminal singularities. Also, "canonical" is a Zariski open condition in families [4]. A pencil of hypersurfaces in $\mathbf{P}^5$ given by the same formula as above shows that rationality does not specialize among 4-folds with canonical singularities. $\square$

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